7.1: Laplace Transforms and Inverse Transforms

Up to now, we have seen how to solve certain types of differential equations in which all the functions are continuous. However, consider an RLC circuit in which we can turn the switch on or off at will. This causes discontinuities in the external force which makes the problem of solving them a more difficult one. Luckily, we have many smart mathematicians before us who have devised clever ways to get around these problems. One specific example is the **Laplace Transform**.

Definition. (The Laplace Transform)//

Given a function $f(t)$ defined for all $t \geq 0$, the Laplace transform of f is the function F defined as follows:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt.$$

for all values of s for which the improper integral converges.

Example 1. Let $f(t) \equiv 1$ for $t \geq 0$. Find $\mathcal{L}\{f(t)\}$.

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} \cdot 1 dt = -\frac{1}{s} e^{-st} \Big|_0^\infty = \frac{1}{s}$$

So $\mathcal{L}\{1\} = \frac{1}{s}$. $(s > 0)$

Example 2. Find the Laplace transform for $f(t) = e^{at}$ for $t \geq 0$.

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{(a-s)t} dt = \frac{1}{s-a}$$

So $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$. $(s > a)$

$\boxed{\text{Gamma Fcn}}$

Example 3. Suppose $f(t) = t^a$ for some $a > -1$. Find $\mathcal{L}\{f(t)\}$.

$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$.

Then $\Gamma(1) = 1$

$\Gamma(x+1) = x\Gamma(x)$

$\Gamma(n+1) = n!$

$$\mathcal{L}\{t^a\} = \int_0^\infty e^{-st} t^a dt \quad \begin{pmatrix} u = st \\ du = s dt \end{pmatrix}$$

$$= \frac{1}{s^{a+1}} \int_0^\infty e^{-u} u^a du$$

$$= \frac{\Gamma(a+1)}{s^{a+1}}$$

Thus $\mathcal{L}\{t\} = \frac{1}{s^2}$, $\mathcal{L}\{t^2\} = \frac{2}{s^3}$, $\mathcal{L}\{t^3\} = \frac{6}{s^4}$

Luckily we do not have to proceed much further strictly using the definition of Laplace transforms. This is a linear transformation! Therefore, just like integration or differentiation, we can start adding/subtracting functions and multiplying constants at will.

Theorem 1. (Linearity of the Laplace Transform)

If a and b are constants, then

$$\mathcal{L}\{af(t)+bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

for all s such that the Laplace transforms of the functions f and g both exist.

Example 4. Find $\mathcal{L}\{t^{n/2}\}$ for $n \geq 1$.

$\Gamma(1/2) = \sqrt{\pi}$. Thus $\mathcal{L}\{t^{1/2}\} = \frac{\sqrt{\pi}}{s^{3/2}}$, $\mathcal{L}\{t^{3/2}\} = \frac{\sqrt{\pi}}{s^{3/2} \cdot 2}$

$\Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2}$

$\Gamma(5/2) = \frac{3}{2}\Gamma(3/2) = \frac{3}{4}\sqrt{\pi}$ $\quad\mathcal{L}\{t^{5/2}\} = \frac{3\sqrt{\pi}}{4s^{7/2}} \ldots$

Example 5. Find $\mathcal{L}\{\cosh kt\}$.

$\mathcal{L}\{\cosh kt\} = \mathcal{L}\left\{\frac{e^{kt}+e^{-kt}}{2}\right\} = \frac{1}{2}\mathcal{L}\{e^{kt}\} + \frac{1}{2}\mathcal{L}\{e^{-kt}\} = \frac{1}{2}\left(\frac{1}{s-k}+\frac{1}{s+k}\right)$

$= \frac{s}{s^2-k^2}\ (s>k>0)$

Do the same for $\cos kt = \frac{e^{ikt}+e^{-ikt}}{2}$, $\sin kt = \frac{e^{ikt}-e^{-ikt}}{2i}$, $\sinh kt = \frac{e^{kt}-e^{-kt}}{2}$.

Exercise 1. Find $\mathcal{L}\{3e^{2t} + 2\sin^2 3t\}$.

$\mathcal{L}\{3e^{2t}+2\sin^2 3t\} = 3\mathcal{L}\{e^{2t}\} + \mathcal{L}\{1\} - \mathcal{L}\{\cos 6t\}$

$= \frac{3}{s-2} + \frac{1}{s} - \frac{s}{s^2+36} = \frac{3s^3+144s-72}{s(s-2)(s^2+36)}\ (s>0)$

$\mathcal{L}\{\sinh kt\} = \frac{k}{s^2-k^2}\ (s>k>0)$, $\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}\ (s>0)$

$\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}\ (s>0)$

The whole goal of the Laplace transform is to change a differential equation that is difficult (or impossible) to solve in the original form to another equivalent differential equation that we can solve easily (enough). Thus, just like every other transformation/substitution that we do, we must be able to transform the solution back to the original. The nice thing is that the Laplace transform is unique for continuous functions. Thus,

$$\text{if } F(s) = \mathcal{L}\{f(t)\}, \quad \text{then } f(t) = \mathcal{L}^{-1}\{F(s)\}.$$

Exercise 2. Find $\mathcal{L}^{-1}\{\frac{1}{s^3}\}$, $\mathcal{L}^{-1}\{\frac{1}{s+2}\}$ and $\mathcal{L}^{-1}\{\frac{2}{s^2+9}\}$.

$\mathcal{L}\{t^2\} = \frac{2}{s^3}$, so $\mathcal{L}^{-1}\{\frac{1}{s^3}\} = \frac{t^2}{2}$.

$\mathcal{L}\{e^{-2t}\} = \frac{1}{s+2}$, so $\mathcal{L}^{-1}\{\frac{1}{s+2}\} = e^{-2t}$

$\mathcal{L}\{\sin 3t\} = \frac{3}{s^2+9}$, so $\mathcal{L}^{-1}\{\frac{2}{s^2+9}\} = \frac{2}{3}\sin 3t$.

Example 6. Let $u_a(t) = \begin{cases} 0 & \text{for } t < a \\ 1 & \text{for } t \geq a \end{cases}$. Find $\mathcal{L}\{u_a(t)\}$ for $a > 0$.

$\mathcal{L}\{u_a(t)\} = \int_0^\infty e^{-st} u_a(t)\,dt = \int_a^\infty e^{-st} = \frac{-e^{-st}}{s}\Big|_a^\infty = \frac{e^{-as}}{s} \quad (s>0)$

Theorem 2. (Existence of Laplace Transform)

If the function f is piecewise continuous for $t \geq 0$ and is of exponential order as $t \to \infty$, then its Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ exists. More precisely, if there exists constants M, c and T such that $|f(t)| \leq Me^{ct}$ for all $t \geq T$, then $F(s)$ exists for all $s > c$.

Corollary ($F(s)$ for large s)

If $f(t)$ satisfies the hypotheses of Theorem 2, then

$$\lim_{s \to \infty} F(s) = 0.$$

Theorem 3. (Uniqueness of Inverse Laplace Transforms)

Suppose that the functions $f(t)$ and $g(t)$ satisfy the hypotheses of Theorem 2, so that their Laplace transforms $F(s)$ and $G(s)$ both exist (on some domain). If $F(s) = G(s)$ for all $s > c$ (for some c), then $f(t) = g(t)$ wherever on $[0, \infty)$ both f and g are continuous.

Homework. 1-31, 35-41